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A note on circles touching two circles in a Pappus chain: Part 3

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Abstract. A similar result to the results in [2, 3, 5] for a circle touching two consecutive circles at their point of tangency in a Pappus chain is given.

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1. Introduction

In [2, 3, 5], we consider a chain of circles whose members touch two given touching circles, and consider a circle touching two consecutive circles in the chain at their point of tangency and the line passing through the centers of the given circles at their point of tangency (see Figures 1 and 2). Then we have shown that a simple relationship between the radius of the circle and the radii of the two given circles using division by zero $1/0 = 0$.

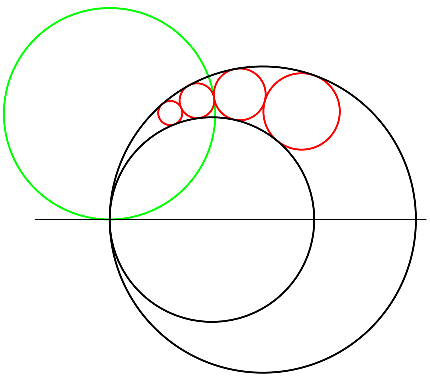

Figure 1.

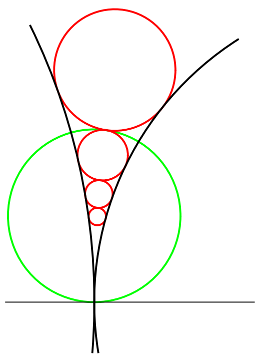

Figure 2.

In this paper we consider a chain of circles whose members touch a given circle and its tangent. Then we will show that a similar relation also holds. We assume division by zero [1], [6]:

$$\frac{z}{0} = 0 \textit{ for any real number } z. \tag{1}$$

2. Result

Let α be a circle of diameter BC such that $|BC| = 2a$. We use a rectangular coordinate system with origin C such that one of the farthest points on α has coordinates (a, a). We use the next theorem.

Theorem 1 ([4], [6])**.** *The following statements holds.*
(i) *A line can be considered to be a circle of radius* 0 *and center at the origin.*
(ii) *Two orthogonal figures can be considered to touch each other.*

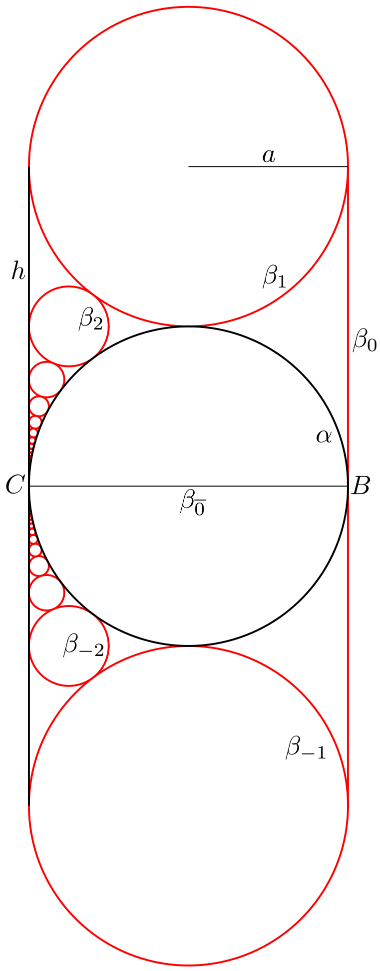

Figure 3.

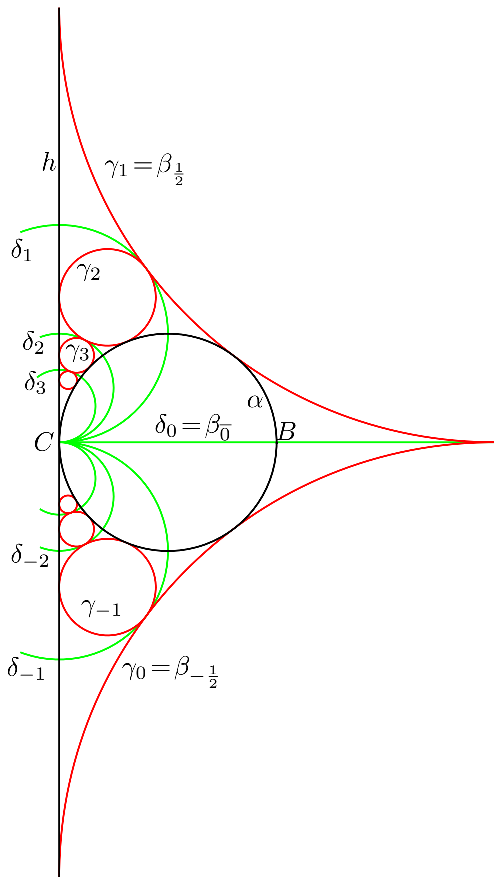

Figure 4.

The tangent of the circle α at the point C is denoted by h (see Figure 3).

Theorem 2. *The following statements holds.*
(i) *A proper circle touches the circle α at a point different from C and the line h if and only if there is a non-zero real number z such that the center and the radius*

of the circle are given by

$$\left(\frac{a}{z^2}, \frac{2a}{z}\right), \quad \frac{a}{z^2}. \tag{2}$$

The circle determined by (2) *is denoted by* β_z.
(ii) *Let* β_0 *be the tangent of* α *at the point* B. *Two circles or a circle and a line* β_w *and* β_z *touch if and only if* $|w - z| = 1$ *for real numbers* w *and* z.
(iii) *Let* b_z *be the radius of the circle* β_z. *Then the distance between the center of* β_z *and the line* BC *equals* $2|z|b_z$.

Proof. If a proper circle has radius and center given by (2), we can easily see that the distance between the centers of this circle and α equals $a + a/z^2$. Hence they touch. Conversely assume that a circle β touches α at a point different from C and h. Then there is a real number z such that the radius of β equals a/z^2. Then the circle, whose radius and center are given by (2), touches α at a point different from C and h as just we have shown. Therefore this circle coincides with β. This proves (i).

The line β_0 touches $\beta_{\pm 1}$ (see Figure 3). If $w \neq 0$ and $z \neq 0$, then β_w and β_z are proper circles and they touch if and only if

$$\sqrt{\left(\frac{a}{w^2} - \frac{a}{z^2}\right)^2 + \left(\frac{2a}{w} - \frac{2a}{z}\right)^2} = \frac{a}{w^2} + \frac{a}{z^2}.$$

While we have

$$\left(\frac{a}{w^2} - \frac{a}{z^2}\right)^2 + \left(\frac{2a}{w} - \frac{2a}{z}\right)^2 - \left(\frac{a}{w^2} + \frac{a}{z^2}\right)^2 = \frac{4a^2((w-z)^2 - 1)}{w^2 z^2}.$$

This proves (ii). The part (iii) follows from (i). □

By (1) and Theorem 1(i), the center and the radius of β_0 are represented by (2) with $z = 0$, and it touches α at point different from C and the line h. However all the lines have center at the origin and radius 0 by Theorem 1(i), i.e., the line having center and radius given by (2) with $z = 0$ is not uniquely determined.

The circles $\beta_{\pm 1/2}$ touch the line BC. Let $\gamma_z = \beta_{z - \frac{1}{2}}$. Then the circles $\gamma_0 = \beta_{-\frac{1}{2}}$ and $\gamma_1 = \beta_{\frac{1}{2}}$ touch BC (see Figure 4). We now show that a similar relation to the results in [2, 3, 5] holds.

Theorem 3. *If* δ_z *is the circle touching the circles* γ_z *and* γ_{z+1} *at their point of tangency and the line* BC *at* C, *then* δ_z *has radius*

$$\frac{a}{|z|},$$

where we define that δ_0 *is the line* BC.

Proof. If we invert the figure in the circle of center C orthogonal to γ_z, then γ_z is fixed, and γ_{z+1} is inverted into the circle congruent to γ_z whose center lies on the perpendicular from the center of γ_z to BC. Hence there is a line parallel to BC touching the images of γ_z and γ_{z+1} at their point of tangency. Inverting this line in the same circle of inversion, we get the circle touching γ_z and γ_{z+1} at their point of tangency and BC at C. This is the circle δ_z. Let d_z be the radius of δ_z. The circle γ_z has center of coordinates $(a/(z - \frac{1}{2})^2, 2a/(z - \frac{1}{2}))$ and radius $a/(z - \frac{1}{2})^2$.

If $z > 0$, then δ_z has center of coordinates $(0, d_z)$ and the circles γ_z and δ_z touch externally. Hence we have

$$\left(\frac{a}{(z-1/2)^2}\right)^2 + \left(\frac{2a}{z-1/2} - d_z\right)^2 = \left(\frac{a}{(z-1/2)^2} + d_z\right)^2.$$

Solving the equation for d_z, we have $d_z = \frac{a}{z}$. If $z < 0$, then δ_z has center of coordinates $(0, -d_z)$, and γ_z and δ_z touch internally. Therefore we have

$$\left(\frac{a}{(z-1/2)^2}\right)^2 + \left(\frac{2a}{z-1/2} + d_z\right)^2 = \left(\frac{a}{(z-1/2)^2} - d_z\right)^2.$$

Solving the equation for d_z, we have $d_z = -\frac{a}{z}$. The theorem is true in the case $z = 0$ by (1) and Theorem 1(i). The proof is now complete. $\square$

We can consider that the line BC touches the circle α at point different from C (which is B) and the line h by Theorem 1(ii). We denote this line by $\beta_{\overline{0}}$, i.e., $\delta_0 = \beta_{\overline{0}}$ (see Figures 3 and 4). The point C can be considered to touch α and h at C, which is the limiting figure when the circles $\beta_{\pm z}$, $\gamma_{\pm z}$ and $\delta_{\pm z}$ approach to C in the case $z \to \infty$. The line h can also be considered to touch α and h.

We have shown that the chain $\cdots$, β_{-1}, β_0, β_1, $\cdots$ has similar properties to those of the chains whose members touch two given touching circles [2, 3, 5]. Therefore it seems to be appropriate to still call this a Pappus chain.

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