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A note on a generalization of a five circle problem: Part 2

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Abstract. We generalize a problem in Wasan geometry involving three smaller congruent circles touching two larger congruent circles and their common tangent.

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1. Introduction

In this note we generalize the following problem in [3] (see Figure 1).

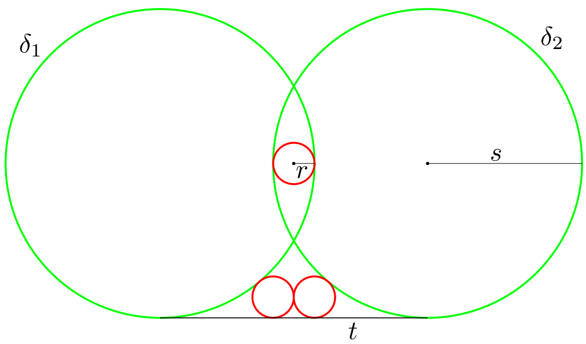

Figure 1: $r = (1 - \sqrt{3/4})s$.

Problem 1. Two intersecting circles δ_1 and δ_2 of radius s have an external common tangent t, and the maximal circle touching the two circles from the inside has radius r. Two touching circles of radius r touch t from the same side as δ_1 and one touches δ_1 externally and the other touches δ_2 externally. Show that $r = (1 - \sqrt{3/4})s$ holds.

Similar problems have considered in [1, 2], which can also be found in [3].

2. Generalization

In this section we generalize the problem. We use the next lemma.

Lemma 1. *For a square $ABCD$ with $|AB| = s$, let δ be a circle of radius s and center A. A circle of radius r touches the segment CD at a point P form the same side as δ and also touches δ externally. Another circle of radius s' touches the segments BC and CP and also touches the circle of radius r externally. If $|CP|/r + 1 = d$, then we have*

$$s = (\sqrt{d} + 1)^2 r \quad \text{and} \quad s' = (\sqrt{d} - 1)^2 r. \tag{1}$$

Proof. From $s = |CP| + 2\sqrt{sr} = (d-1)r + 2\sqrt{sr}$, we have $s = (\sqrt{d} \pm 1)^2 r$ (see Figure 2). Similarly from $s' = (d-1)r - 2\sqrt{s'r}$, we have $s' = (\sqrt{d} \pm 1)^2 r$. Since $s > s'$, we have (1). □

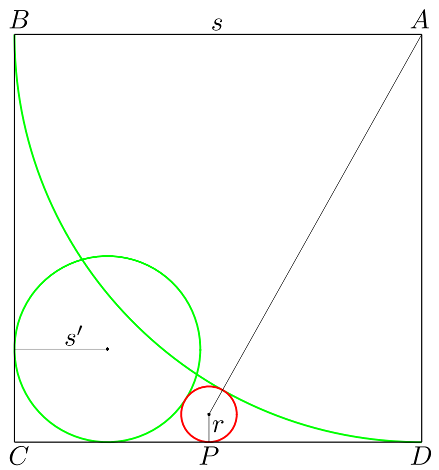

Figure 2.

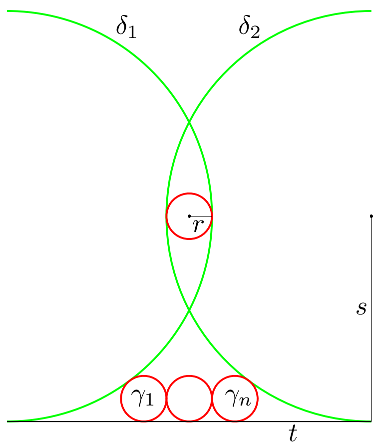

Figure 3.

If $\gamma_1, \gamma_2, \cdots, \gamma_n$ are congruent circles such that γ_1 and γ_2 touch, and γ_i touches γ_{i-1} at the farthest point on γ_{i-1} from γ_1 for $i = 3, 4, \cdots, n$, then the circles are said to be *congruent circles in line.* The problem is generalized as follows:

Theorem 1. *Two intersecting circles δ_1 and δ_2 of radius s have an external common tangent t, and the maximal circle touching the two circles from the inside has radius r. $\gamma_1, \gamma_2, \cdots, \gamma_n$ are congruent circles of radius r in line lying inside of the curvilinear triangle made by δ_1, δ_2 and t such that they touch t and γ_1 touches δ_1 and γ_n touches δ_2. Then the following statements are true.*
(i) $s = (\sqrt{n+1} + 1)^2 r$.
(ii) *s/r is an integer if and only if there is an integer $j > 1$ such that $n = (j-1)(j+1)$. In this event $s/r = (j+1)^2$.*
(iii) *If n is an odd integer such that $n = 2k - 1$ for a positive integer k, then there are circles $\gamma'_1, \gamma'_2, \cdots, \gamma'_k$ of radius r in line such that $\gamma'_k = \gamma_k$ and γ'_1 touches δ_1 and δ_2 externally.*

Proof. The distance between the centers of γ_1 and γ_n equals $2(n-1)r$ (see Figure 3). Therefore the distance between the center of γ_n and the perpendicular to t touching δ_2 and the maximal circle touching δ_1 and δ_2 equals $(n-1)r + r = nr$. Therefore (i) is proved by Lemma 1. The part (ii) is obvious. Let ρ be the reflection in the line joining the centers of δ_1 and γ_k, and let $\gamma'_i = \gamma_i^\rho$ for $i = 1, 2, \cdots, k$.

Then γ'_1, γ'_2, $\cdots$, γ'_k form congruent circles in line. Since the circles δ_1 and γ_k are fixed by ρ, the circle γ'_1 touches δ_1 (see Figure 4). It also touches δ_2 by the symmetry of the figure. This proves (iii). □

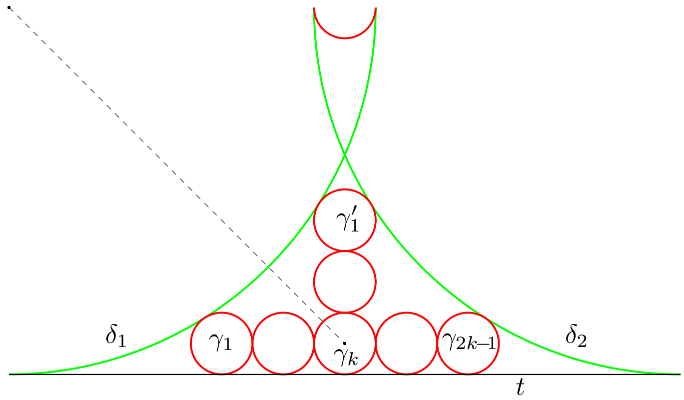

Figure 4: $k = 3$.

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[3] No author name, Enri Shinjutsu (圓理新術), no date, Digital Library Department of Mathematics Kyoto University.