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The arbelos in Wasan geometry: Totsuka's problem (Problem 2017-3-6)

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Abstract. We generalize Totsuka's problem involving a collinear arbelos, which is also proposed as Problem 6 in [12].

Keywords. arbelos, collinear arbelos, congruent circles in line.

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1. Introduction

Let δ_1, δ_2, $\cdots$, δ_n be congruent circles with collinear centers such that δ_1 and δ_2 touch and $\delta_i (\neq \delta_{i-2})$ touches δ_{i-1} for $i = 3, 4, \cdots, n$. In this case we call δ_1, δ_2, $\cdots$, δ_n congruent circles in line (see Figure 1).

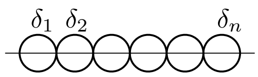

Figure 1.

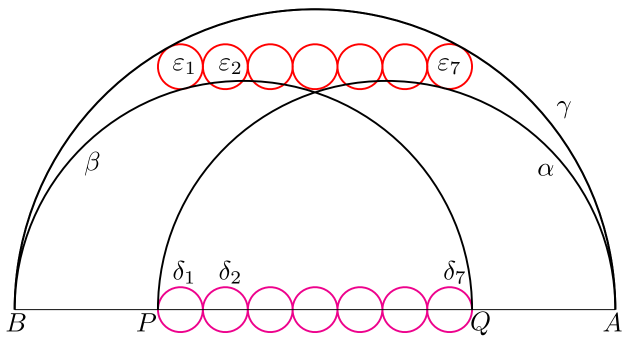

Figure 2: H_7

In this article we consider the following configuration: For two points P and Q on the segment AB symmetric about the midpoint of AB, let α, β and γ be the semicircles of diameters AP, BQ and AB, respectively constructed on the same side of AB. The configuration of the three semicircles is a special case of a generalized arbelos called the collinear arbelos [6], [9]. Let δ_1, δ_2, $\cdots$, δ_n be congruent circles in line of radius r such that δ_1 touches α internally at P and δ_n touches β internally at Q. Then we assume that there are congruent circles ε_1, ε_2, $\cdots$, ε_n in line such that they have radius r and ε_1 touches β externally and γ internally and ε_n touches α externally and γ internally. In this case the configuration consisting of the three semicircles and the pair of the congruent circles in line is denoted by H_n (see Figure 2). If $P = Q$, we denote the configuration of the three semicircles by H_0. The radii of α and γ are denoted by s and t, respectively for H_n, where we define $r = s$ if $n = 0$.

The next problem for H_2 using Figure 3 was proposed by Totsuka (戸塚古羽) in 1820 [2].

Problem 1. Show $t = 6r$ for H_2.

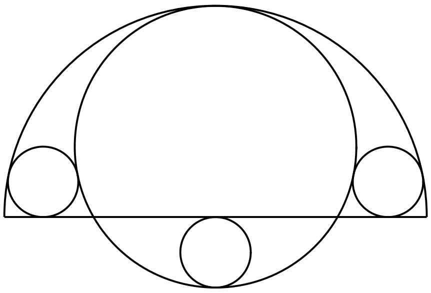

Figure 3.

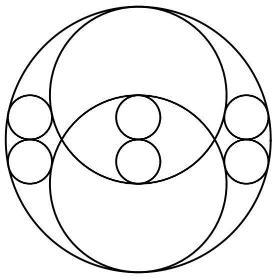

Figure 4.

The same problem using the same figure can be found in [3, 10, 11, 14, with no date]. Essentially the same problem using Figure 4 is proposed as Problem 6 in [12], which is taken from[5][2]. Similar problems for H_1 can be found in [4, dated 1800] and [1, 13, 15, with no date].

2. GENERALIZATION

We generalize Problem 1. We use a rectangular coordinate system with origin at the midpoint of AB so that the point A has coordinates $(t, 0)$ and the farthest point on γ from AB has coordinates $(0, t)$. We denote the coordinates of the center of the circle ε_i $(1 \leq i \leq n)$ by (x_i, y_e) for H_n. The next proposition gives a generalization of Problem 1.

Proposition 1. *The following relations hold for H_n.*

$$s = \frac{2n + 1 + \sqrt{4n+1}}{2} r, \qquad t = (n + 1 + \sqrt{4n+1})r. \tag{1}$$

$$x_i = (2i - n - 1)r, \qquad y_e = \sqrt{2n\left(3 + \sqrt{4n+1}\right)}\, r. \tag{2}$$

Proof. The center of β has x-coordinate $s - t$, and we have $x_1 = -(n-1)r$. Solving the equations $x_1^2 + y_e^2 = (t - r)^2$, $(x_1 - (s - t))^2 + y_e^2 = (r + s)^2$ and $|AB| = 2t = 4s - 2nr$ for s, t and y_e, we have (1) and the last part of (2). Obviously we get $x_i = x_1 + 2(i - 1)r = (2i - n - 1)r$. □

If $n = 0$, then $t = 2r = 2s$ by (1). Therefore we get $P = Q$ (see Figure 5, where the circle η_1 will be explained later), and the circle ε_i does not exist. Therefore the proposition holds in this case. Assume $n = 1$. The proposition shows that $s = \phi^2 r$ for H_1, where $\phi = (1 + \sqrt{5})/2$. Hence $2s/|BP| = s/(s - r) = \phi$. Therefore

[2]The links of Wasan books in the references in [12] are dead due to the URL changes.

the semicircle of diameter BP constructed on the same side of AB as α, and α and γ form a golden arbelos for H_1, where the circle ε_1 is one of the twin circles of Archimedes (see Figure 6).

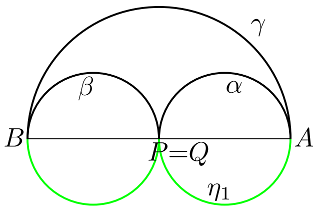

Figure 5: $n = 0$, H_0.

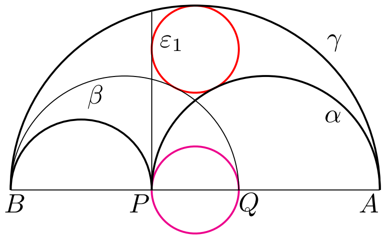

Figure 6: H_1.

We show that if O and S are the centers of γ and α, respectively, then $2|OS| = |AQ|$ holds. Let p and q be the x-coordinates of the points P and Q, respectively. Then $p + q = 0$. While $|OS| = (t + p)/2$. Therefore we have $2|OS| = t + p = t - q = |AQ|$. The relation holds even if α and β have no point in common.

3. Integer case

In this section we consider the case in which the ratio t/r is an integer for H_n.

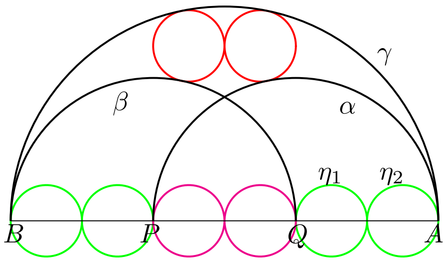

Figure 7: $k = 1$, H_2.

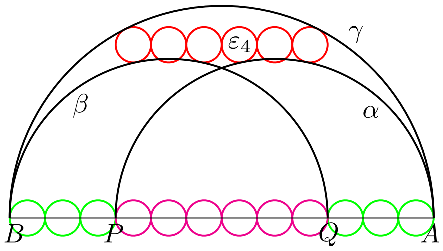

Figure 8: $k = 2$, H_6, ε_4 touches α.

Theorem 1. *The ratio t/r is an integer if and only if there is a non-negative integer k such that $n = k(k + 1)$ for H_n. In this event, we have the followings.*
(i) *The following relations hold.*

$$s = (k + 1)^2 r, \quad t = (k + 1)(k + 2)r, \tag{3}$$

$$x_i = (2i - k(k + 1) - 1)r, \quad y_e = 2\sqrt{k(k + 1)(k + 2)}r. \tag{4}$$

(ii) *There are $k + 1$ congruent circles $\eta_1, \eta_2, \cdots, \eta_{k+1}$ in line such that they have radius r and η_1 touches β externally at Q and η_{k+1} touches α internally at A.*

Proof. The ratio t/r is an integer if and only if $4n + 1$ is the square of an integer by (1), which is equivalent to $4n + 1 = (2k + 1)^2$ for a non-negative integer k. The last equation is equivalent to $n = k(k + 1)$ and (3) and (4) hold in this event by (1) and (2). The part (ii) is proved by the following relation obtained by (3) :

$$|AQ| = |BP| = t - nr = 2(k + 1)r.$$

□

For a similar result for Problem 2017-3-5 see [7] and [8]. If $k = 0$, then the half part of η_1 coincides with α (see Figure 5), but the circle ε_i does not exist. Therefore Theorem 1 holds in the case $k = 0$. The fact justifies our definition $r = s$ in the case $n = 0$. Problem 1 is the case $k = 1$ (see Figure 7).

Assume $k \geq 1$. The point P has x-coordinate $-k(k+1)r$, while the point A has x-coordinate $(k+1)(k+2)r$ by (3). Hence the point of intersection of α and β, which coincides with the point of intersection of α and the y-axis, has y-coordinate $(k+1)\sqrt{k(k+2)}r$. Therefore the center of ε_1 lies on the line passing through this point parallel to AB if and only if $k = 3$ by (4). While H_n is symmetric about the y-axis. Hence the point of tangency of the circles ε_6 and ε_7 lie on the y-axis for H_{12}, i.e., they touch at the point of intersection of α and β (see Figure 9).

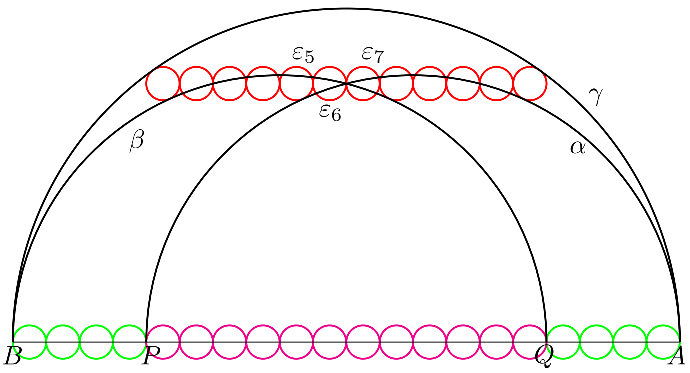

Figure 9: $k = 3$, H_{12}, ε_6 and ε_7 touch at the point of intersection of α and β.

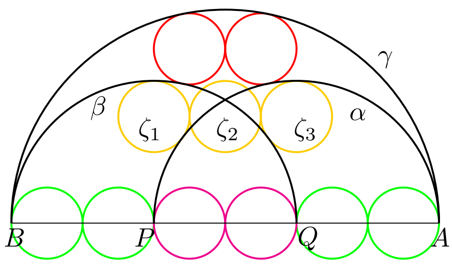

Figure 10: $k = 1$.

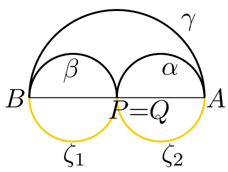

Figure 11: $k = 0$.

Theorem 2. *The following statements hold for* $H_{k(k+1)}$.
(i) *The circle* ε_{k+2} *touches* α *externally.*
(ii) *There are* $k+2$ *congruent circles* $\zeta_1, \zeta_2, \cdots, \zeta_{k+2}$ *in line such that they have radius* r *and* ζ_1 *touches* α *externally and* β *internally, and* ζ_{k+2} *touches* α *internally and* β *externally.*
(iii) *The orthogonal projection of the centers of* ζ_1 *and* ζ_{k+2} *to* AB *coincide with the centers of* β *and* α*, respectively.*

Proof. Recall that the centers of ε_i and α have coordinates (x_i, y_e) and $(x_a, 0) = (t-s, 0) = ((k+1)r, 0)$ by (3), respectively. Then ε_i and α touch externally if and only if $(x_a - x_i)^2 + (0 - y_e)^2 - (s+r)^2 = 0$. The left side of the equation equals $4(k+2-i)(k(k+1)-i)r^2$ by (3) and (4). This proves (i) (see Figures 8 and 9). Let $\zeta_1, \zeta_2, \cdots, \zeta_i$ be congruent circles in line and let (x_z, y_z) be the coordinates of the center of ζ_1. Then ζ_1 touches α externally and β internally, and ζ_i touches α internally and β externally if and only if $(x_z-(k+1)r)^2+y_z^2 = (s+r)^2$, $(x_z+(k+1)r)^2+y_z^2 = (s-r)^2$ and $(x_z+2(i-1)r-(k+1)r)^2+y_z^2 = (s-r)^2$. Solving the equations for i, x_z and y_z, we have $i = k+2$ and $(x_z, y_z) = (-(k+1)r, k(k+2)r)$. This proves (ii) and (iii) (see Figure 10). □

The circle ε_{k+2} does not exist if $k = 0, 1$, i.e., (i) holds in this case. If $k = 0$, then the half parts of the circles ζ_1 and $\zeta_2 = \eta_1$ coincide with β and α, respectively (see Figure 11).

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