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A note on a problem involving a square in a curvilinear triangle

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Abstract. A problem involving a square in the curvilinear triangle made by two congruent touching circles and their common tangent is generalized.

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1. Introduction

Let α_1 and α_2 be touching circles of radius a with external common tangent t. In this note we generalize the following problem in Wasan geometry [1, 5, 6] (see Figure 1).

Problem 1. Let $ABCD$ be a square such that the side DA lies on t and the points C and B lie on α_1 and α_2, respectively. Show that $2a = 5|AB|$.

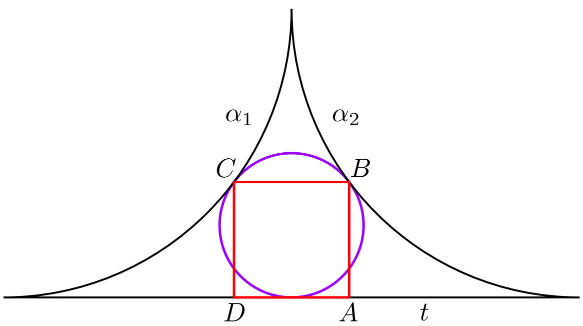

Figure 1.

2. Generalization

We use the following theorem, which is stated as a result for an Archimedean circle of the arbelos denoted by W_5 in [2].

Theorem 1. *Let α and β be externally touching circles of radii a and b, respectively, with external common tangent s. If d is the distance between s and the point of tangency of α and β, the following relation holds.*

$$d = \frac{2ab}{a+b}. \tag{1}$$

If $\gamma_1, \gamma_2, \cdots, \gamma_n$ are congruent circles touching a line s from the same side such that γ_1 and γ_2 touch and γ_i $(i = 3, 4, \cdots, n)$ touches γ_{i-1} from the side opposite to γ_1, then $\gamma_1, \gamma_2, \cdots, \gamma_n$ are called congruent circles on s. We denote the curvilinear triangle made by α_1, α_2 and t by Δ. The incircle of Δ touches α_1 and α_2 at C and B, respectively as shown in Figure 1. Indeed the problem is generalized as follows (see Figure 2):

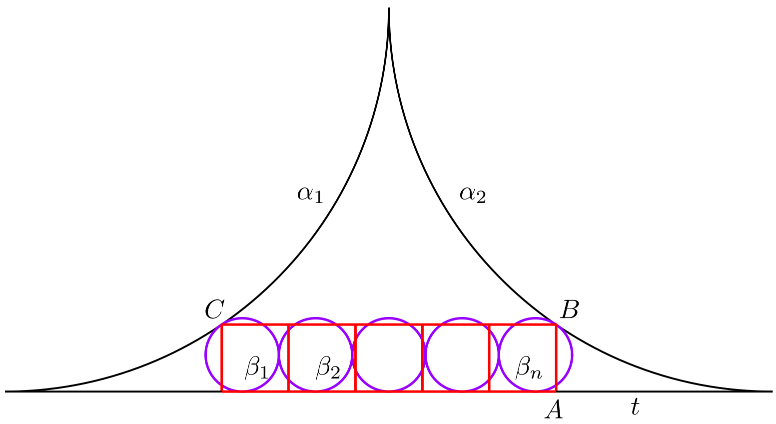

Figure 2: $n = 5$

Theorem 2. *If $\beta_1, \beta_2, \cdots, \beta_n$ are congruent circles on the line t of radius b lying in Δ such that β_1 touches α_1 at a point C and β_n touches α_2 at a point B and A is the foot of perpendicular from B to t, then the following relations hold.*
(i) $n|AB| = |BC|$.
(ii) $\dfrac{2a}{|AB|} = 1 + (\sqrt{n}+1)^2$.

Proof. By Theorem 5.1 in [3] we have $a = (\sqrt{n}+1)^2 b$. Let $d = |AB|$. By (1)

$$\frac{2a}{d} = 1 + \frac{a}{b} = 1 + (\sqrt{n}+1)^2.$$

This proves (ii). Let $|BC| = 2h$. Then from the right triangle formed by the line BC, the segment joining the centers of α_1 and β_1, and the perpendicular from the center of α_1 to BC, we get $(a-h)^2 + (a-d)^2 = a^2$. Solving the equation for h, we have $h = a - \sqrt{(2a-d)d} = an/(1 + (\sqrt{n}+1)^2)$. This proves (i) by (ii). □

The figure consisting of α_1, α_2, β_1, β_2, $\cdots$, β_n and t is denoted by $\mathcal{B}(n)$ and considered in [3]. Since the inradius of Δ equals $a/4$, the next theorem is also a generalization of Problem 1 (see Figure 3).

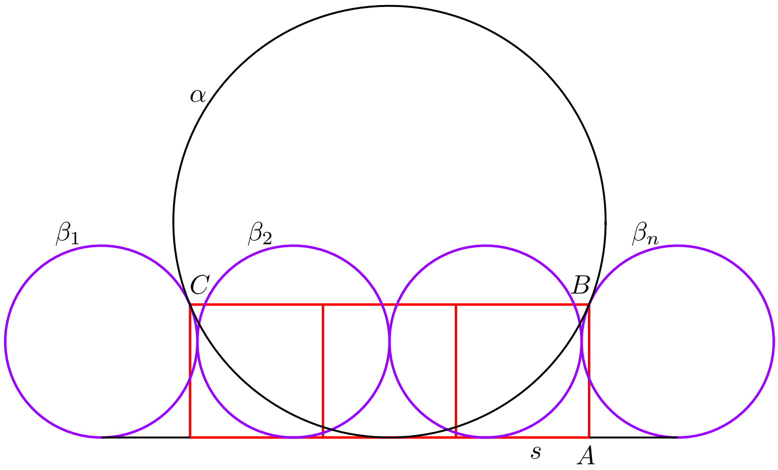

Figure 3: $n = 4$

Theorem 3. *Let β_1, β_2, $\cdots$, β_n be congruent circles on a line s of radius b. If a circle α of radius a touches s and β_1 and β_n externally at points C and B, respectively, A is the foot of perpendicular from B to s, then the following relations hold.*
(i) $(n-1)|AB| = |BC|$.
(ii) $\dfrac{2a}{|AB|} = 1 + \left(\dfrac{n-1}{2}\right)^2$.

Theorem 3 is proved in a similar way as Theorem 2 using the fact

$$\frac{a}{b} = \left(\frac{n-1}{2}\right)^2$$

[4]. The figure consisting of α, β_1, β_2, $\cdots$, β_n and s is denoted by $\mathcal{A}(n)$ and considered in [3].

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