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Theorems on two congruent circles on a line

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Abstract. Problems involving two congruent circles on a line are generalized.

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1. Introduction and Preliminaries

Two externally touching congruent circles with common external tangent s are called two congruent circles on a line or two congruent circles on s. Let α and β be externally touching circles of radii a and b, respectively with external common tangent s. In this note, we consider two congruent circles on s such that they touch s from the same side as α, one of which touches α externally and the other touches β externally. If each of ρ_1, ρ_2, ρ_3 is a circle or a line and they form a curvilinear triangle, the triangle and its incircle are denoted by $T(\rho_1, \rho_2, \rho_3)$ and $I(\rho_1, \rho_2, \rho_3)$, respectively. The following problem can be found in [1], [2], [3], [4, 5], [6], [8], [9] (see Figure 1).

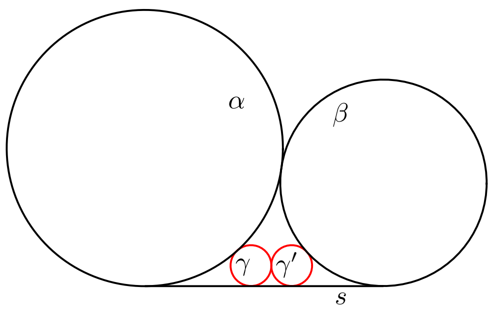

Figure 1.

Problem 1. *Let γ and γ' be two congruent circles on s of radius c such that they lie in $T(\alpha, \beta, s)$, and γ touches α and γ' touches β. Find c in terms of a and b.*

We consider the problem in a general way. We use the next proposition.

Proposition 1.1. *The following statements hold.*
(i) *If s touches α and β at points P and Q, $|PQ| = 2\sqrt{ab}$.*
(ii) *If c is the radius of $I(\alpha, \beta, s)$, then*

$$\frac{1}{\sqrt{c}} = \frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}}. \tag{1}$$

2. Main results

We get the following theorem (see Figure 2).

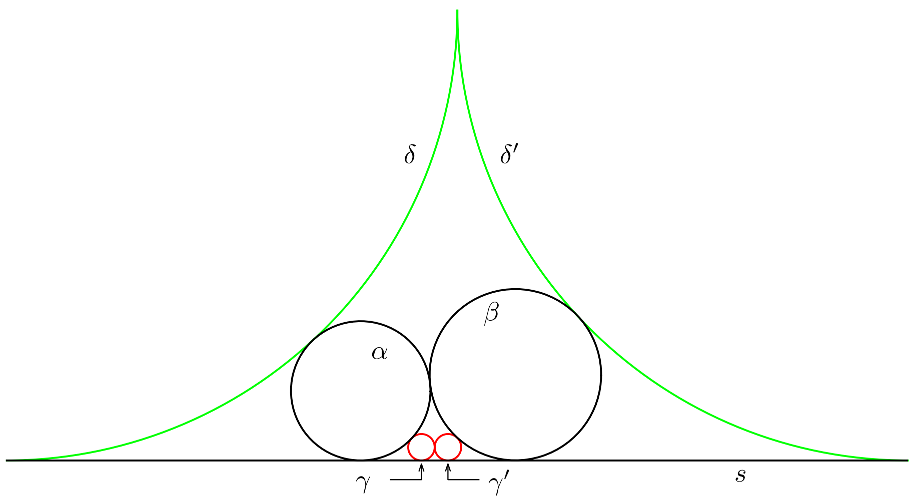

Figure 2

Theorem 2.1. *Let γ and γ' (resp. δ and δ') be two congruent circles on s of radius c (resp. d) such that α and β lie in $T(\delta, \delta', s)$, γ and γ' lie in $T(\alpha, \beta, s)$, γ and δ touch α externally and γ' and δ' touch β externally. The following relations hold.*
(i) $\sqrt{a} + \sqrt{b} = \sqrt{d} - \sqrt{c}$.
(ii) $c = \dfrac{w - \sqrt{w^2 - 4ab}}{2}$ *and* $d = \dfrac{w + \sqrt{w^2 - 4ab}}{2}$, *where* $w = a + b + 4\sqrt{ab}$.
(iii) $ab = cd$.

Proof. By Proposition 1.1(i), we get

$$2\sqrt{ab} = 2\sqrt{ac} + 2\sqrt{bc} + 2c \tag{2}$$

and

$$2\sqrt{ab} + 2\sqrt{ad} + 2\sqrt{bd} = 2d. \tag{3}$$

Eliminating $\sqrt{ab}$ from (2) and (3), we get $\left(\sqrt{a} + \sqrt{b}\right)\left(\sqrt{c} + \sqrt{d}\right) = d - c$. This proves (i). Solving (2) and (3) for c and d, we have $c = \left(w \pm \sqrt{w^2 - 4ab}\right)/2$ and $d = \left(w \pm \sqrt{w^2 - 4ab}\right)/2$. This proves (ii). The part (iii) follows from (ii). □

Problems asking to find the relation (i) of the next theorem can be found in [7] and [10].

Theorem 2.2. *Assume that γ, γ', δ, δ' are as in Theorem 2.1. If e and e' are the radii of the circles $\varepsilon = I(\alpha, \gamma, s)$ and $\varepsilon' = I(\beta, \gamma', s)$, respectively, also f and f' are the radii of the circles $\zeta = I(\alpha, \delta, s)$ and $\zeta' = I(\beta, \delta', s)$, respectively, then*

the following relations hold.
(i) $c^2 = 4ee'$. (ii) $ab = 4ff'$.

Proof. By Proposition 1.1(ii) we get

$$\frac{1}{\sqrt{e}} = \frac{1}{\sqrt{a}} + \frac{1}{\sqrt{c}} \quad \text{and} \quad \frac{1}{\sqrt{e'}} = \frac{1}{\sqrt{b}} + \frac{1}{\sqrt{c}}$$

(see Figure 3). Therefore by Theorem 2.1(i), (iii) we get

$$\begin{aligned}\frac{1}{\sqrt{ee'}} &= \left(\frac{1}{\sqrt{a}} + \frac{1}{\sqrt{c}}\right)\left(\frac{1}{\sqrt{b}} + \frac{1}{\sqrt{c}}\right) = \frac{c + \sqrt{c}(\sqrt{a}+\sqrt{b}) + \sqrt{ab}}{c\sqrt{ab}} \\ &= \frac{c + \sqrt{c}\left(\sqrt{a}+\sqrt{b}\right) + \sqrt{cd}}{c\sqrt{cd}} = \frac{\sqrt{c}+\sqrt{a}+\sqrt{b}+\sqrt{d}}{c\sqrt{d}} = \frac{2\sqrt{d}}{c\sqrt{d}} = \frac{2}{c}.\end{aligned}$$

This proves (i). Similarly from

$$\frac{1}{\sqrt{f}} = \frac{1}{\sqrt{a}} + \frac{1}{\sqrt{d}} \quad \text{and} \quad \frac{1}{\sqrt{f'}} = \frac{1}{\sqrt{b}} + \frac{1}{\sqrt{d}},$$

we get

$$\begin{aligned}\frac{1}{\sqrt{ff'}} &= \left(\frac{1}{\sqrt{a}} + \frac{1}{\sqrt{d}}\right)\left(\frac{1}{\sqrt{b}} + \frac{1}{\sqrt{d}}\right) = \frac{d + \sqrt{d}(\sqrt{a}+\sqrt{b}) + \sqrt{ab}}{d\sqrt{ab}} \\ &= \frac{d + \sqrt{d}(\sqrt{a}+\sqrt{b}) + \sqrt{cd}}{d\sqrt{ab}} = \frac{\sqrt{d}+\sqrt{a}+\sqrt{b}+\sqrt{c}}{\sqrt{abd}} = \frac{2\sqrt{d}}{\sqrt{abd}} = \frac{2}{\sqrt{ab}}.\end{aligned}$$

This proves (ii). □

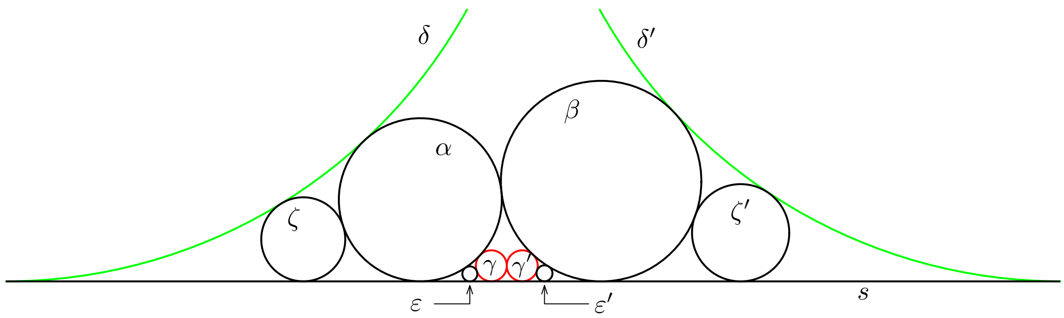

Figure 3

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Tohoku Univ. WDB is short for Tohoku University Wasan Material Database.